L¹-smoothing for the Ornstein-Uhlenbeck semigroup

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May 24, 2012

Abstract

Given a probability density, we estimate the rate of decay of the measure of the level sets of its evolutes by the Ornstein-Uhlenbeck semigroup. It is faster than what follows from the preservation of mass and Markov’s inequality.

1 Introduction

Let N ≥ 1. For t ≥ 0, consider the probability measure µt = 1 − e−t t−1 + 1+e−t t1. We simply write µ for µ∞ = 1(δ−1 + δ1). On the (multiplicative) group {−1, 1}N, we consider the semigroup of operators (Tt)t≥0 defined for functions f : {−1, 1}N → R by

Ttf = f * µtN.

In other words,

Ttf(x) = ∫ f(x · y)Kt(y) dµN(y),

where Kt(y) = N ∏i=1 (1 + e−tyi). For A ⊂ {1, . . . , N}, we define wA : {−1, 1}N → R by wA(y) = ∏i∈A yi with the convention w∅ = 1. This family, known as the Walsh system, forms an orthonormal basis of L²({−1, 1}N, µN). Expanding the product in the definition of the kernel Kt one readily checks that TtwA = e−t card(A)wA.

The above formulations show that T0 ◦ Tt = Tt+1, that Tt is self-adjoint on L² and preserves positivity and integrals (with respect to µN). As a consequence Tt is a contraction from Lp = Lp({−1, 1}N, µN) into itself: ∥Ttf∥p ≤ ∥f∥p for p ≥ 1. Actually, the hypercontractive estimate of Bonami [2] and Beckner [1] tells us more: if 1 < p < q < +∞ and e2t ≥ 2−1−1 p−1, then

∥Ttf∥q ≤ ∥f∥p.

Hence the semigroup improves the integrability of functions in Lp provided p > 1. A challenging problem is to understand the improving effects of Tt on functions f ∈ L¹. In the paper [5], Talagrand asks the following question: for t > 0, is there a function ψt : [1, +∞) → (0, +∞) with limu→+∞ ψt(u) = +∞, such that for every N ≥ 1 and every function f on {−1, 1}N with ∥f∥1 ≤ 1, and all u > 1,

μN ( {x : |Ttf(x)| > u} ) ≤ 1 ψt(u) ?

(1)

This would be a strong improvement on the following simple consequence of Markov’s inequality and the contractivity property:

μN ( {x : |Ttf(x)| > u} ) ≤ ∥Ttf∥1 u ≤ ∥f∥1 u.

Talagrand actually asks a more specific question with ψt(u) = ct(t)√log(u) and he observes that one cannot expect a faster rate in u. Question (1) is still open; only in some special cases is an affirmative answer known (see the last section). Its difficulty is essentially due to the lack of convexity of the tail

*Research of the third, fourth and fifth named authors was partially supported by Polish MNiSzW grants 1 PO3A 012 29 and N N201 397437
condition. Nevertheless, the paper [5] contains a similar result for the averaged operator $M := \int_0^1 T_t \, dt$: there exists $K$ such that for all $N$ and $u > 1$,

$$\mu^N \{ \{x : |Mf(x)| \geq u\|f\|_1\} \} \leq K \frac{\log \log u}{u \log u}.$$  

The goal of this note is to study the analogue of Question (1) in Gauss space.

## 2 Gaussian setting

Let $n \geq 1$. We work on $\mathbb{R}^n$ with its canonical Euclidean structure $(\langle \cdot, \cdot \rangle, | \cdot |)$. Denote by $\gamma_n$ the standard Gaussian probability measure on $\mathbb{R}^n$:

$$\gamma_n(dx) = e^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$$

Let $G$ be a standard Gaussian random vector, with distribution $\gamma_n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then the Ornstein-Uhlenbeck semigroup $(U_t)_{t \geq 0}$ is defined by

$$U_tf(x) = \mathcal{E}f(e^{-t}x + \sqrt{1-e^{-2t}}G)$$

$$= \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y) e^{-|y|^2/2} \frac{dy}{(2\pi)^{n/2}}$$

$$= (1-e^{-2t})^{-n/2} \int_{\mathbb{R}^n} f(z) e^{-\frac{|z|^2}{2(1-e^{-2t})}} \frac{dz}{(2\pi)^{n/2}}$$

$$= (1-e^{-2t})^{-n/2} e^{|x|^2/2} \int_{\mathbb{R}^n} f(z) e^{-\frac{|x-z|^2}{2(1-e^{-2t})}} d\gamma_n(z),$$

when $f$ is nonnegative or belongs to $L^1(\gamma_n)$. The operators $U_t$ preserve positivity and mean. They are self-adjoint on $L^2(\gamma_n)$. By Nelson’s hypercontractivity theorem [3], $U_t$ is a contraction from $L^p(\gamma_n)$ to $L^q(\gamma_n)$ provided $1 < p \leq q$ and $(p-1)e^{2t} \geq q - 1$. It is natural to ask the analogue of Question (1) for $U_t$: does there exist a function $\psi_t$ with $\lim_{u \to +\infty} \psi_t(u) = +\infty$ such that for all $n$ and all nonnegative or $\gamma_n$-integrable function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\gamma_n \left\{ \{x : |U_tf(x)| > u\|f\|_{L^1(\gamma_n)}\} \right\} \leq \frac{1}{u \psi_t(u)}? \tag{2}$$

This inequality would actually follow from Talagrand’s conjecture on the discrete cube. Indeed, if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and bounded, consider the function $g : \{-1,1\}^{nk} \to \mathbb{R}$ defined by

$$g((x_{i,j})_{1 \leq i \leq n, j \leq k}) = f \left( \frac{x_{1,1} + \cdots + x_{1,k}}{\sqrt{k}}, \ldots, \frac{x_{n,1} + \cdots + x_{n,k}}{\sqrt{k}} \right).$$

By the Central Limit Theorem, when $k$ goes to infinity, the distribution of $g$ under $\mu^{nk}$ tends to that of $f$ under $\gamma_n$, while the distribution of $T_2g$ under $\mu^{nk}$ tends to that of $U_2f$ under $\gamma_n$ (see e.g. [1]). This allows us to pass from (1) for $g$ to (2) for $f$. The above argument uses boundedness and continuity. These assumptions can be removed by a classical truncation argument, and using the semigroup property: $U_tf = U_{t/2}U_{t/2}f$ where $U_{t/2}f$ is automatically continuous. We omit the details.

To conclude this introduction, let us provide evidence that the functions $\psi_t(u)$ in (2) cannot grow faster than $\sqrt{\log u}$. We will do this for $n = 1$, which implies the general case (by choosing functions depending on only one variable). We have showed that

$$U_tf(x) = \int_{\mathbb{R}} Q_t(x,z)f(z) d\gamma_1(z), \tag{3}$$

where

$$Q_t(x,z) = (1-e^{-2t})^{-\frac{1}{2}} \exp \left( \frac{1}{2} \left( x^2 - \frac{(z-e^{t}x)^2}{e^{2t}-1} \right) \right).$$
We are going to choose specific functions \( f \geq 0 \) with \( \int f d\gamma_1 = 1 \) for which \( U_t f \) can be explicitly computed. Note that (3) allows us to extend the definition of \( U_t \) to (nonnegative) measures \( \nu \) with \( \int \varphi d\nu < +\infty \) where \( \varphi(t) = e^{-t^2/2}/\sqrt{2\pi} \) is the Gaussian density. The simplest choice is then to take normalized Dirac masses \( \tilde{\delta}_y := \varphi(y)^{-1}\delta_y \) as test measures. Obviously \( \int \varphi d\tilde{\delta}_y = 1 \) and \( U_t \tilde{\delta}_y = Q_t(\cdot, y) \).

Actually, by the semigroup property, \( Q_t(\cdot, y) = U_{t\gamma_2}U_t\tilde{\delta}_y = U_{t\gamma_2}Q_t(\cdot, y) \), where \( x \mapsto Q_t(\cdot, y) \) is a nonnegative function with unit Gaussian integral. Hence,

\[
\{Q_t(\cdot, y); \ y \in \mathbb{R}\} \subseteq \left\{U_{t\gamma_2} f : f \geq 0 \text{ and } \int f d\gamma_1 = 1 \right\}.
\]

Fix \( t > 0 \) and let \( u > (1 - e^{-2t})^{-1/2} \). Then using \( Q_t(x, y) = Q_t(y, x) \) and setting \( v = u\sqrt{1 - e^{-2t}} \) one readily gets that

\[
\{x : Q_t(x, y) > u\} = \left\{x : \exp\left(\frac{1}{2}(y^2 - (x - e^t y)^2)\right) > v\right\} = \left\{e^t y - \sqrt{(e^{2t} - 1)(y^2 - 2 \log v)^+} : e^t y + \sqrt{(e^{2t} - 1)(y^2 - 2 \log v)^+}\right\}.
\]

For the particular choice \( y = y_0 := e^t \sqrt{2 \log v} \), one gets

\[
\{x : Q_t(x, y_0) > u\} = \left(\sqrt{2 \log v}; (2e^{2t} - 1)\sqrt{2 \log v}\right).
\]

Since for \( 0 < a < b \), \( \gamma_1((a, b)) \geq \int_{b}^{b + e^{-s^2/2}ds/\sqrt{2\pi}} = \frac{e^{-s^2/2} - e^{-s^2/2}}{b\sqrt{2\pi}} \), we can deduce that

\[
\gamma_1(\{x : Q_t(x, y_0) > u\}) \geq \frac{1}{2\sqrt{\pi}(2e^{2t} - 1)\sqrt{\log v}} \left(\frac{1}{v} - \frac{1}{(2e^{2t} - 1)^2}\right).
\]

Combining the above observations yields

\[
\liminf_{u \to +\infty} u\sqrt{\log u} \sup \left\{\gamma_1(\{x : U_{t\gamma_2} f(x) > u\}) : f \geq 0 \text{ and } \int f d\gamma_1 = 1 \right\} > 0.
\]

Hence \( \psi_t(u) \) in (2) cannot grow faster than \( \sqrt{\log u} \).

Using the same one-dimensional test functions and similar calculations, one can check that for \( t > 0 \), the image by \( U_t \) of the unit ball \( B_1 = \{f \in L^1(\gamma_n) : ||f||_1 \leq 1\} \) is not uniformly integrable, that is:

\[
\liminf_{|c| \to 1} \sup_{f \in B_1} \int_{|f| > c} |f| d\gamma_n > 0.
\]

Consequently \( U_t : L^1(\gamma_n) \to L^\infty(\gamma_n) \) is not continuous when \( \phi \) is a Young function with \( \lim_{t \to +\infty} \phi(t)/t = +\infty \). Next, we turn to positive results.

## 3 Main results

In the rest of this section \( B(a, r) \) denotes the closed ball of center \( a \) and radius \( r \), while \( C(a, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 \leq |x - a| \leq r_2\} \). We start with an easy inclusion of the upper level-sets of \( U_t f \).

**Lemma 1.** Let \( f : \mathbb{R}^n \to \mathbb{R}^+ \) be such that \( \int f d\gamma_n = 1 \). Then for all \( t, u > 0 \),

\[
\left\{x \in \mathbb{R}^n : U_t f(x) > u\right\} \subseteq B\left(0, \sqrt{\left(2 \log u + n \log(1 - e^{-2t})\right)^+}\right).
\]

**Proof.** As already explained

\[
U_t f(x) = (1 - e^{-2t})^{-n/2} e^{x^2/2} \int_{\mathbb{R}^n} f(z) e^{-\frac{e^{-2t}}{2(1 - e^{-2t})^2}(z - e^t x)^2} d\gamma_n(z).
\]

Consequently \( U_t f(x) \leq (1 - e^{-2t})^{-n/2} e^{x^2/2} \int d\gamma_n \). Our normalization hypothesis then implies that

\[
\left\{x : U_t f(x) > u\right\} \subseteq \left\{x : |x|^2 > 2 \log u + n \log(1 - e^{-2t})\right\}.
\]

\( \square \)

3
The probability measure of complements of balls appearing in the above lemma can be estimated thanks to the following classical fact.

**Lemma 2.** For all $n \in \mathbb{N}^*$ there exists a constant $c_n$ such that for all $u \geq \sqrt{2n}$ it holds
\[ \gamma_n(B(0,u)^c) \leq c_n u^{n-2} e^{-u^2/2}. \]

Actually, when $n \leq 2$ this is valid for all $u > 0$. Also one may take $c_1 = \sqrt{2/\pi}$.

**Proof.** Polar integration gives that
\[ \gamma_n(B(0,u)^c) = (2\pi)^{-n/2} \cdot n \ vol_n(B(0,1)) \int_u^{+\infty} r^{n-1} e^{-r^2/2} \ dr. \]

For $u^2 \geq 2n - 4$ the map $r \mapsto r^{n-2} e^{-r^2/4}$ is non-increasing on $(u, \infty)$. Thus we may bound the last integral:
\[ \int_u^{+\infty} r^{n-2} e^{-r^2/4} \cdot r e^{-r^2/4} \ dr \leq \int_u^{+\infty} u^{n-2} e^{-u^2/4} \cdot u e^{-u^2/4} \ dr = 2u^{n-2} e^{-u^2/2}. \]

Combining the previous statements gives a satisfactory estimate in dimension $1$, which improves on the Markov estimate $\gamma_n(U_t f \geq u) \leq \min(1,1/u)$ if $f$ is non-negative with integral $1$.

**Proposition 3.** Let $f : \mathbb{R} \to \mathbb{R}^+$ be integrable. Then for all $t > 0$ and $v > 1$,
\[ \gamma_1\left(\left\{x : U_t f(x) > v \frac{\int f \, d\gamma_1}{\sqrt{1-e^{-v^2}}}\right\}\right) \leq \frac{1}{v \sqrt{\pi} \log v}. \]

In higher dimension, the above reasoning gives a weaker estimate than Markov’s inequality. However a more precise approach allows us to get a slightly weaker decay for the level sets of $U_t f$. Our main result is stated next. It contains a dimensional dependence that we were not able to remove.

**Theorem 4.** Let $n \geq 2$ and $t > 0$. Then there exists a constant $K(n,t)$ such that for all non-negative functions $f$ defined on $\mathbb{R}^n$ with $\int f \, d\gamma_n = 1$, for all $u > 10$,
\[ \gamma_n\left(\left\{x \in \mathbb{R}^n : U_t f(x) > u\right\}\right) \leq K(n,t) \frac{\log \log u}{u \sqrt{\log u}}. \]

**Proof.** Note that it is enough to show the inequality for $u$ larger than some number $u_0(n,t) > 10$ depending only of $n$ and $t$. We will just write that we choose $u$ large enough, but an explicit value of $u_0(n,t)$ can be obtained from our argument. Let us define
\[ R_1 = R_1(u,n,t) := \left(2 \log u + n \log(1 - e^{-2t})\right)^{\frac{1}{2}}, \]
\[ R_2 = R_2(u,n) := (2 \log u + (n-1) \log \log u)^{\frac{1}{2}}. \]

It is clear that for $u$ large enough $R_2 > \sqrt{2n}$ and also $R_2 > R_1 > 0$. So by Lemma 2,
\[ \gamma_n(B(0,R_2)^c) \leq c_n e^{-R_2^2/2} R_2^{n-2} = c_n \frac{(2 \log u + (n-1) \log \log u)^{\frac{n-2}{2}}}{\left(\log u\right)^{\frac{n-2}{2}}} \]
\[ \leq c_n \left(\frac{(n+1) \log u}{\left(\log u\right)^{\frac{n+1}{2}}} \right)^{\frac{n-2}{2}} = c'_n \frac{(n+1) \log u}{u \sqrt{\log u}}. \]
By Lemma 1, \( \{ x : U_t f(x) > u \} \) is a subset of \( B(0, R_1)^c \). Hence we may write, using Markov’s inequality on the annulus \( C(0, R_1, R_2) \) and the self-adjointness of \( U_t \):

\[
\gamma_n(\{ x : U_t f(x) > u \}) \leq \gamma_n(\{ x : U_t f(x) > u \} \cap B(0, R_2)) + \gamma_n(B(0, R_2)^c)
\]

\[
= \gamma_n(\{ x : U_t f(x) > u \} \cap B(0, R_1)^c \cap B(0, R_2)) + \gamma_n(B(0, R_2)^c)
\]

\[
\leq \int \frac{U_t f}{u} \mathbf{1}_{C(0, R_1, R_2)^c} d\gamma_n + \gamma_n(B(0, R_2)^c)
\]

\[
= \frac{1}{u} \int (U_t \mathbf{1}_{C(0, R_1, R_2)}) f d\gamma_n + \gamma_n(B(0, R_2)^c)
\]

\[
\leq \frac{1}{u} \| U_t \mathbf{1}_{C(0, R_1, R_2)} \|_{\infty} + \frac{c_n}{u \sqrt{\log u}}.
\]

To prove the theorem, it remains to show that \( \| U_t \mathbf{1}_{C(0, R_1, R_2)} \|_{\infty} = O\left( \frac{\log \log u}{\sqrt{\log u}} \right) \). First note that for any set \( A \subset \mathbb{R}^n \) and all \( x \in \mathbb{R}^n \),

\[
U_t \mathbf{1}_A(x) = E \mathbf{1}_A(e^{-t}x + \sqrt{1 - e^{-2t}} G) = P\left( G \in \frac{A - e^{-t}x}{\sqrt{1 - e^{-2t}}} \right) = \gamma_n\left( \frac{A - e^{-t}x}{\sqrt{1 - e^{-2t}}} \right).
\]

Therefore

\[
\| U_t \mathbf{1}_{C(0, R_1, R_2)} \|_{\infty} = \sup_{a \in \mathbb{R}^n} \gamma_n(C(a, \tilde{R}_1, \tilde{R}_2)),
\]

where \( \tilde{R}_i := R_i/\sqrt{1 - e^{-2t}} \). The main idea is that the shells described above can be covered by a thin slab and the complement of a large ball. Set

\[
r = r(u) := 2(\log \log u)^{\frac{1}{2}},
\]

then for \( u \) large enough, Lemma 2 yields

\[
\gamma_n(B(0, r)^c) \leq c_n e^{-r^2/2} u^{-2} = c_n 2^{n-2} \left( \frac{\log \log u}{\log u} \right)^{n-2} \leq c_n \left( \frac{\log \log u}{\sqrt{\log u}} \right).
\]

For an arbitrary point \( a \in \mathbb{R}^n \),

\[
\gamma_n(C(a, \tilde{R}_1, \tilde{R}_2)) \leq \gamma_n(C(a, \tilde{R}_1, \tilde{R}_2) \cap B(0, r)) + \gamma_n(B(0, r)^c)
\]

\[
\leq \gamma_n(C(a, \tilde{R}_1, \tilde{R}_2) \cap B(0, r)) + c_n \frac{\log \log u}{\sqrt{\log u}}
\]

For \( u \) large enough, \( r < \tilde{R}_1 \), and Lemma 5 which is stated below ensures that \( C(a, \tilde{R}_1, \tilde{R}_2) \cap B(0, r) \) is contained in a strip \( S \) of width

\[
w := \tilde{R}_2 - \sqrt{\tilde{R}_1^2 - r^2}.
\]

By the product properties of the Gaussian measure, \( \gamma_n(S) \) coincides with the one-dimensional Gaussian measure of an interval of length \( w \). Therefore it is not bigger than \( w/\sqrt{2\pi} \leq w \). Hence

\[
\gamma_n(C(a, \tilde{R}_1, \tilde{R}_2) \cap B(0, r)) \leq \tilde{R}_2 - \sqrt{\tilde{R}_1^2 - r^2}
\]

\[
\leq \sqrt{2 \log u + (n-1) \log \log u} - \frac{2 \log u + n \log(1 - e^{-2t}) - 4 \log \log u}{1 - e^{-2t}}
\]

\[
\leq \frac{(n-1 + 4(1 - e^{-2t})) \log u - n \log(1 - e^{-2t})}{\sqrt{1 - e^{-2t}} \sqrt{2 \log u + (n-1) \log \log u}} \leq \kappa(n, t) \frac{\log \log u}{\sqrt{\log u}},
\]

where the last inequality is valid for \( u \) large enough. The proof of the theorem is therefore complete. \( \square \)

\textbf{Lemma 5.} Let \( 0 < r < \rho_1 < \rho_2 \) and \( a, b \in \mathbb{R}^n \), then the set

\[
C(a, \rho_1, \rho_2) \cap B(b, r)
\]

is contained in a strip of width at most \( \rho_2 - \sqrt{\rho_1^2 - r^2} \).
Proof. Assume that the intersection is not empty. Then without loss of generality, $a = 0$ and $b = te_1$ with $t > 0$. Let $z$ be an arbitrary point in the intersection. Obviously $z_1 \leq |z| \leq \rho_2$. Next, since $z \in B(b,r)$ and $|z| \geq \rho_1$, one gets

$$r^2 \geq |z-te_1|^2 = |z|^2 - 2tz_1 + t^2 \geq \rho_1^2 - 2tz_1 + t^2.$$ 

Hence by the arithmetic mean-geometric mean inequality the first coordinate of $z$ satisfies

$$z_1 \geq \frac{1}{2} \left( \frac{\rho_1^2 - r^2}{t} + t \right) \geq \sqrt{\rho_1^2 - r^2}.$$ 

Summarizing, $z \in \left[ \sqrt{\rho_1^2 - r^2}, \rho_2 \right] \times \mathbb{R}^{n-1}$. □

4 Product functions on the discrete cube

Finally, we provide an affirmative answer to the Question (1) in the case of functions with product structure.

**Proposition 6.** Assume that functions $f_1, f_2, \ldots, f_N : \{-1,1\} \rightarrow [0, \infty)$ satisfy $\int f_i d\mu = 1$ for $i = 1, 2, \ldots, N$. Let $f = f_1 \otimes f_2 \otimes \ldots \otimes f_N$, i.e. $f(x) = \prod_{i=1}^{N} f_i(x_i)$. Then for every $t > 0$ there exists a positive constant $c_t$ such that for all $u > 1$

$$\mu^N \left( \{ x : |T_t f(x)| > u \} \right) \leq \frac{c_t}{u \sqrt{\log u}}.$$ 

**Proof.** The above result is immediately implied by the following inequality.

**Proposition 7.** ([4]) Let $b > a > 0$. Let $X_1, X_2, \ldots, X_N$ be independent non-negative random variables such that $EX_i = 1$ and $a \leq X_i \leq b$ a.s. for $i = 1, 2, \ldots, N$. Then for every $u > 1$ we have

$$P \left( \prod_{i=1}^{N} X_i > u \right) \leq Cu^{-1}(1 + \log u)^{-1/2},$$ 

where $C$ is a positive constant which depends only on $a$ and $b$.

Indeed, $T_t f = T_t f_1 \otimes T_t f_2 \otimes \ldots \otimes T_t f_N$, where $T_t f_i : \{-1,1\} \rightarrow [1 - e^{-t}, 1 + e^{-t}]$ satisfy $\int T_t f_i d\mu = \int f_i d\mu = 1$ for $i = 1, 2, \ldots, N$. Thus random variables $X_1, X_2, \ldots, X_N$ defined on the probability space $(\{-1,1\}^N, \mu^N)$ by $X_i(x) = T_t f_i(x_i)$ satisfy assumptions of Proposition 7 with $a = 1 - e^{-t}$ and $b = 1 + e^{-t}$ while $f = \prod_{i=1}^{N} X_i$. □

References


